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# Note on graviton MHV amplitudes

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Abstract: Two new formulas which express $n$-graviton MHV tree amplitudes in terms of sums of squares of $n$-gluon amplitudes are discussed. The first formula is derived from recursion relations. The second formula, simpler because it involves fewer permutations, is obtained from the variant of the Berends, Giele, Kuijf formula given in Arxiv:0707.1035.

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## Contents



## 1. Introduction

Spinor-helicity methods have been used in work on gauge theories for many years. Spinor expressions for S-matrix elements are usually much simpler than the sum of contributing Feynman diagrams as in the strikingly simple Parke-Taylor [1] formula for color ordered maximal helicity violating (MHV) gluon amplitudes in tree approximation:

$$A_n(1^-, 2^-, 3^+, \ldots, n^+) = \frac{\langle 1\,2 \rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle}\,. \tag{1.1}$$

The bracket $\langle j\,k\rangle = -\langle k\,j\rangle$ is the invariant product of positive helicity spinor solutions of the massless Dirac equation for particles of 4-momentum $p_j^\mu$ and $p_k^\mu$. Much information about the formalism can be found in reviews such as [2–4]. The subject was reinvigorated by the use of twistor ideas [5] which led to recursion relations [6] for tree amplitudes in which the spinors are treated as complex variables. Feynman diagram computations can be replaced by the algebraic process of solving the recursion relations.

Recursion relations have also been derived for tree approximation graviton amplitudes [7, 8], and these are an important ingredient of this paper. MHV amplitudes describe processes involving two negative and $(n-2)$ positive helicity particles. It is well known that these are simpler in both gauge theory and gravity than non-MHV amplitudes which have more than two negative helicity particles. Our primary concern is the set of MHV graviton amplitudes $M_n(1^-, 2^-, 3^+, \ldots, n^+)$.

Our interest in this subject was motivated by recent papers in which the 3-loop graviton 4-point function was calculated in $\mathcal{N} = 8$ supergravity and shown to be ultraviolet finite [9, 10]. The structures found in the calculation (and in earlier work cited in these papers) led the authors to speculate that the S-matrix of $\mathcal{N} = 8$ supergravity is ultraviolet finite to all orders of perturbation theory. In the computational approach used in this program loop amplitudes are constructed from tree amplitudes by studying unitarity cuts. Thus tree approximation amplitudes are a basic ingredient of higher loop calculations and simplified expressions for tree amplitudes can be useful.

The well known KLT relations [11] express graviton tree amplitudes $M_n$ in terms of products $A_n\,A'_n$ of gluon amplitudes in which the momenta in $A'_n$ are a permutation of those of $A_n$. The KLT relations for $n = 4$ and $n = 5$ external lines are

$$M_4(1,2,3,4) \;=\; -s_{12}\,A_4(1,2,3,4)\,A_4(1,2,4,3)\,, \tag{1.2}$$

$$M_5(1,2,3,4,5) \;=\; s_{23}\,s_{45}\,A_5(1,2,3,4,5)\,A_5(1,3,2,5,4) + (3 \leftrightarrow 4)\,. \tag{1.3}$$

The formulas are more complicated for general $n$. (See appendix A of [12].) The KLT relations are valid for all helicity configurations, and similar formulas relate amplitudes for any choice of particles in supergravity to products of amplitudes in supersymmetric gauge theory. In particular tree amplitudes in $\mathcal{N} = 8$ supergravity are related to products of amplitudes for $\mathcal{N} = 4$ gauge theory.

The KLT relations were obtained from string theory. From the perspective of field theory, however, the relations are very surprising. The Lagrangian of Yang-Mills theory, with 3- and 4-point vertices only, appears to be far simpler than the Einstein-Hilbert Lagrangian, which contains complicated $n$-point two-derivative interactions. While the 4-point KLT relation has been derived directly from graviton Feynman rules [13], and field redefinitions have been explored [14, 15], no general field theory derivation has been given.

The work presented here is a modest step towards such a derivation and toward the goal of simplified amplitudes. We present two formulas for $n$-graviton MHV amplitudes, each of which expresses $M_n(1^-,2^-,3^+,\ldots,n^+)$ as a sum of terms containing squares $A_n(1^-,2^-,i_3^+,\ldots,i_n^+)^2$ of gluon amplitudes, where $i_3,\ldots,i_n$ indicates a permutation of the positive helicity lines. The first formula is derived from recursion relations. The complicated structure of the Lagrangian is thus avoided, but field theoretic properties such as analyticity and factorization underlie the recursion relations, and the on-shell 3-graviton vertex is required. The second formula is obtained by manipulation of a recently presented version [16] of the BGK formula [17].

The formula derived from recursion relations is (for $n \geq 4$)

$$M_n(1^-,2^-,3^+,\ldots,n^+) = \sum_{\mathcal{P}(i_3,\ldots,i_n)} s_{1i_n} \left(\prod_{s=4}^{n-1} \beta_s\right) A_n(1^-,2^-,i_3^+,\ldots,i_n^+)^2\,, \tag{1.4}$$

where[1]

$$\beta_s = -\frac{\langle i_s\, i_{s+1}\rangle}{\langle 2\, i_{s+1}\rangle}\,\langle 2|\, i_3 + i_4 + \cdots + i_{s-1} |i_s]\,. \tag{1.5}$$

[1]The notation includes spinors $j]$, $k]$ which are negative helicity solutions of the Dirac equation for null momenta $p_j^\mu$, $p_k^\mu$. They appear through $[j\,k]$ and $\langle j|i|k]$ which are defined by ($p_i^\mu$ is also null)

$$[j\,k] \;=\; \frac{s_{jk}}{\langle k\,j\rangle} \;=\; -\,\frac{(p_j+p_k)^2}{\langle k\,j\rangle}$$
$$\langle j|\,i\,|k] \;=\; \langle j|\not{p}_i\,|k] \;=\; \langle j\,i\rangle[i\,k]\,.$$



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The sum in (1.4) is over all permutations $\mathcal{P}(i_3,\ldots,i_n)$ of the external positive helicity labels $\{3,4,\ldots,n\}$. Our new version of the BGK formula is

$$M_n(1^-,2^-,3^+,\ldots,n^+)=\sum_{\mathcal{P}(i_4,\ldots,i_n)}\frac{\langle 12\rangle\langle i_3 i_4\rangle}{\langle 1 i_3\rangle\langle 2 i_4\rangle}s_{1i_n}\left(\prod_{s=4}^{n-1}\beta_s\right)A_n(1^-,2^-,i_3^+,\ldots,i_n^+)^2, \quad (1.6)$$

with the same $\beta_s$. The distinguished line $i_3$ can be any chosen member of the set $\{3,4,\ldots,n\}$, and the sum includes all permutations of the remaining $n-3$ members.

The evidence that the formulas above are correct includes:

(i) analytic proof that (1.4) agrees for all $n$ with the MHV formula given in [7].

(ii) Analytic proof for $n=4$, 5 that both (1.4) and (1.6) agree, and also agree with the KLT results (1.2)–(1.3).

(iii) Numerical work showing that (1.4) agrees with the original BGK formula [17] for all $n\leq 12$.

(iv) Numerical tests of the agreement between (1.4) and (1.6) for all $n\leq 12$ and additional tests that different choices of $i_3$ in (1.6) do not change the result.

The derivation of (1.4) follows the approach of [7] to recursion relations, but we organize permutations differently and use gauge theory recursion relations to simplify the work and the result. This is presented in section 2. In section 3 the passage from the BGK formula of [16] to (1.6) is outlined. It would be interesting and useful to extend the treatment of recursion relations to non-MHV amplitudes, but this is much more difficult. Our progress here is limited to a formula for the anti-MHV 5-point function $M_5(1^-,2^-,3^-,4^+,5^+)$ presented in section 4.

## 2. Derivation of MHV formula (1.4)

The simple elegant theory underlying recursion relations has been described clearly in [6–8], so we dispense with the background and start with the elements we need. Recursion relations require a shift of either the $|\,j\,]$ or $|\,j\,\rangle$ spinor of a pair of momenta in $n$-point tree amplitudes. We follow [7] and use a $[2,1\rangle$-shift, i.e.

$$|\hat{1}\rangle=|1\rangle-z|2\rangle\,,\qquad |\hat{1}]=|1]\,,\qquad |\hat{2}]=|2]+z|1]\,,\qquad |\hat{2}\rangle=|2\rangle\,. \quad (2.1)$$

Recursion relations are valid if the analytically continued amplitude vanishes at large $z$, and this property holds for $(--)$ shifts for gluons [6] and for MHV gravitons [18].

With this choice, the gluon and graviton MHV recursion relations become particularly simple. The gluon recursion relation contains the single term

$$A_n(1^-,2^-,3^+,\ldots,n^+)=A_3(\hat{1}^-,-P^+_{\hat{1}n},n^+)\,\frac{1}{s_{1n}}\,A_{n-1}(P^-_{\hat{1}n},\hat{2}^-,3^+,\ldots,(n-1)^+)\,, \quad (2.2)$$

since color order must be preserved. The graviton recursion relation

$$M_n(1^-,2^-,3^+,\ldots,n^+) = \sum_{\mathcal{P}_c(i_3,\ldots,i_n)} M_3(\hat{1}^-,-P^+_{\hat{1}i_n},i_n^+)\,\frac{1}{s_{1i_n}}\,M_{n-1}(P^-_{\hat{1}i_n},\hat{2}^-,i_3^+,\ldots,i^+_{n-1}) \quad (2.3)$$

contains one term for each of the positive helicity lines. (The sum is over the *cyclic* permutations of these lines.)

In the recursion relations (2.2)–(2.3) each term is evaluated at the value of $z$ that takes the shifted momentum $P^\mu_{\hat{1}k}$ on-shell. Hence

$$0 \;=\; P^2_{\hat{1}k} \;=\; \langle\hat{1}\,k\rangle[1\,k] \;=\; \big(\langle 1\,k\rangle - z\langle 2\,k\rangle\big)\,[1\,k]\,, \quad (2.4)$$

determines the value

$$z = \frac{\langle 1k\rangle}{\langle 2k\rangle}\,. \quad (2.5)$$

The formula (1.4) can be established by an inductive argument using the fact that $M_3$ and $A_3$ are simply related by

$$M_3(1^-,-P^+_{\hat{1}k},j^+) = A_3(1^-,-P^+_{\hat{1}k},j^+)^2. \quad (2.6)$$

The basis of induction is established by showing that our formula reproduces the KLT result for $n=4$. This is done at the end of the section. We assume that (1.4) holds for $M_n$, and then use the recursion relation for $M_{n+1}$ as follows:

$$\begin{aligned} &M_{n+1}\big(1^-,2^-,3^+,\ldots,(n+1)^+\big)\\ &\qquad = \frac{1}{(n-2)!}\sum_{\mathcal{P}(i_3,\ldots,i_{n+1})} M_3(\hat{1}^-,-P^+_{\hat{1}i_{n+1}},i^+_{n+1})\,\frac{1}{s_{1i_{n+1}}}\,M_n(P^-_{\hat{1}i_{n+1}},\hat{2}^-,i_3^+,\ldots,i_n^+)\,. \end{aligned}$$

Bose symmetry of $M_n$ under exchange of any two positive helicity lines was used to turn the sum over cyclic permutations in (2.3) into a sum over all permutations. The factor $1/(n-2)!$ compensates the overcounting.

The formula (1.4) is now substituted for the $n$-point graviton amplitude, and (2.6) is used to write $M_3 = A_3^2$. Then

$$\begin{aligned} &M_{n+1}\big(1^-,2^-,3^+,\ldots,(n+1)^+\big)\\ &\quad = \frac{1}{(n-2)!}\sum_{\mathcal{P}(i_3,\ldots,i_{n+1})} A_3(\hat{1}^-,-P^+_{\hat{1}i_{n+1}},i^+_{n+1})^2\,\frac{1}{s_{1i_{n+1}}}\\ &\qquad\times \sum_{\mathcal{P}(i_3,\ldots,i_n)} s_{i_n P_{\hat{1}i_{n+1}}}\left(\prod_{s=4}^{n-1}\beta_s\right) A_n(P^-_{\hat{1}i_{n+1}},\hat{2}^-,i_3^+,\ldots,i_n^+)^2\\ &\quad = \sum_{\mathcal{P}(i_3,\ldots,i_{n+1})} A_3(\hat{1}^-,-P^+_{\hat{1}i_{n+1}},i^+_{n+1})^2\frac{1}{s_{1i_{n+1}}}s_{i_n P_{\hat{1}i_{n+1}}}\left(\prod_{s=4}^{n-1}\beta_s\right) A_n(P^-_{\hat{1}i_{n+1}},\hat{2}^-,i_3^+,\ldots,i_n^+)^2\\ &\quad = \sum_{\mathcal{P}(i_3,\ldots,i_{n+1})} s_{1i_{n+1}}\,s_{i_n P_{\hat{1}i_{n+1}}}\left(\prod_{s=4}^{n-1}\beta_s\right) A_{n+1}\big(1^-,2^-,i_3^+,\ldots,i^+_{n+1}\big)^2\,. \end{aligned} \quad (2.7)$$



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The factor $1/(n-2)!$ cancels because of the redundant inner permutation sum. In the last line we use the gauge theory recursion relation (2.2) to replace the product $A_3 A_n$ by $s A_{n+1}$.

The final step in the proof is to show that $s_{i_n P_{\hat{1} i_{n+1}}} = \beta_n$. Recall that $P^\mu_{\hat{1} i_{n+1}}$ is a null vector with $z$ evaluated as in (2.5), i.e. $z = \langle 1\, i_{n+1}\rangle / \langle 2\, i_{n+1}\rangle$. Then, using that $P^2_{\hat{1} i_{n+1}} = 0$, we have

$$\begin{aligned}
s_{i_n P_{\hat{1} i_{n+1}}} &= -\Big(p_{i_n} + (p_{\hat{1}} + p_{i_{n+1}})\Big)^2 \\
&= -2 p_{i_n} \cdot p_{\hat{1}} - 2 p_{i_{n+1}} \cdot p_{i_n} \\
&= -\langle \hat{1} i_n\rangle [\hat{1} i_n] - \langle i_{n+1} i_n\rangle [i_{n+1} i_n] \\
&= -\frac{[1\, i_n]}{\langle 2\, i_{n+1}\rangle} \Big( \langle 1\, i_n\rangle \langle 2\, i_{n+1}\rangle - \langle 2\, i_n\rangle \langle 1\, i_{n+1}\rangle \Big) - \langle i_{n+1}\, i_n\rangle [i_{n+1}\, i_n] \\
&= \frac{\langle i_n\, i_{n+1}\rangle}{\langle 2\, i_{n+1}\rangle} \Big( \langle 21\rangle [1\, i_n] + \langle 2\, i_{n+1}\rangle [i_{n+1}\, i_n] \Big) = \frac{\langle i_n\, i_{n+1}\rangle}{\langle 2\, i_{n+1}\rangle} \langle 2|\, 1 + i_{n+1} | i_n] \\
&= -\frac{\langle i_n\, i_{n+1}\rangle}{\langle 2\, i_{n+1}\rangle} \langle 2|\, i_3 + i_4 + \cdots + i_{n-1} | i_n] = \beta_n \,.
\end{aligned} \tag{2.8}$$

We used the Schouten identity in the 5th line and momentum conservation in the last step. This establishes (1.4) for $M_{n+1}$, and the inductive proof is complete.

Let's examine the cases $n = 4, 5$ of (1.4) in more detail. For $n = 4$, the product in (1.4) is over the empty set and is set equal to 1. One then finds

$$M_4(1^-, 2^-, 3^+, 4^+) \;=\; s_{14}\, A_4(1^-, 2^-, 3^+, 4^+)^2 + (3 \leftrightarrow 4)\,. \tag{2.9}$$

Using the explicit form of gluon tree amplitudes (1.1) one can show (using momentum conservation) that $A_4(1^-, 2^-, 3^+, 4^+)$ differs from $A_4(1^-, 2^-, 4^+, 3^+)$ differ by a simple factor of $s_{13}/s_{14}$, and hence (2.9) gives

$$M_4(1^-, 2^-, 3^+, 4^+) = \left( s_{14}\, \frac{s_{13}}{s_{14}} + s_{13}\, \frac{s_{14}}{s_{13}} \right) A_4(1^-, 2^-, 3^+, 4^+)\, A_4(1^-, 2^-, 4^+, 3^+)\,. \tag{2.10}$$

The KLT result (1.2) then follows from $s_{12} + s_{13} + s_{14} = 0$.

For $n = 5$,

$$\prod_{s=4}^{n-1} \beta_s \;=\; \beta_4 \;\;=\;\; -\frac{\langle i_4\, i_5\rangle}{\langle 2\, i_5\rangle} \langle 2|\, i_3 | i_4] \;\;=\;\; -\frac{\langle i_4\, i_5\rangle}{\langle 2\, i_5\rangle} \langle 2\, i_3\rangle [i_3\, i_4]\,. \tag{2.11}$$

Using this one can show analytically that (1.4) reproduces the KLT result (1.3).

### 2.1 Connection to the graviton MHV formula of [7]

The result of [7] for MHV graviton amplitudes is

$$\begin{aligned}
&M_n(1^-, 2^-, 3^+, \ldots, n^+) \\
&\quad = (-1)^{n+1} \sum_{\mathcal{P}(i_3, \ldots, i_n)} \frac{\langle 12\rangle^6 [1 i_n]}{\langle 1 i_n\rangle} \frac{1}{2} \frac{[i_3 i_4]}{\langle 2 i_3\rangle \langle 2 i_4\rangle \langle i_3 i_4\rangle \langle i_3 i_5\rangle \langle i_4 i_5\rangle} \left( \prod_{s=5}^{n-1} \frac{\langle 2| i_3 + \cdots + i_{s-1} | i_s]}{\langle 2 i_{s+1}\rangle \langle i_s i_{s+1}\rangle} \right).
\end{aligned} \tag{2.12}$$

It is not difficult to obtain (2.12) from (1.4). We write the gauge theory MHV amplitude as

$$A_n\big(1^-,2^-,i_3^+,\ldots,i_n^+\big) = \frac{\langle 1\,2\rangle^3}{\langle 2\,i_3\rangle\langle i_3\,i_4\rangle\Big(\prod_{s=4}^{n-1}\langle i_s\,i_{s+1}\rangle\Big)\langle i_n\,1\rangle}\,. \tag{2.13}$$

Substitute this into the MHV relation (1.4) and use $s_{1i_n} = -\langle 1\,i_n\rangle[1\,i_n]$. Then

$$\begin{aligned} &M_n(1^-,2^-,3^+,\ldots,n^+) \\ &\quad= (-1)^n \sum_{\mathcal{P}(i_3,\ldots,i_n)} s_{1i_n}\left(\prod_{s=4}^{n-1}\frac{\langle i_s\,i_{s+1}\rangle}{\langle 2\,i_{s+1}\rangle}\,\langle 2|\,i_3+\cdots+i_{s-1}|i_s]\right) A_n(1^-,2^-,i_3^+,\ldots,i_n^+)^2 \\ &\quad= (-1)^{n+1}\sum_{\mathcal{P}(i_3,\ldots,i_n)}\frac{\langle 1\,2\rangle^6\,[1\,i_n]}{\langle 2\,i_3\rangle^2\langle i_3\,i_4\rangle^2\langle 1\,i_n\rangle}\,\frac{\langle 2|\,i_3|i_4]}{\langle 2\,i_5\rangle\langle i_4\,i_5\rangle}\left(\prod_{s=5}^{n-1}\frac{\langle 2|\,i_3+\cdots+i_{s-1}|i_s]}{\langle 2\,i_{s+1}\rangle\langle i_s\,i_{s+1}\rangle}\right). \end{aligned} \tag{2.14}$$

Using that $\langle 2|\,i_3|i_4] = \langle 2\,i_3\rangle[i_3\,i_4]$, we find

$$\begin{aligned} &M_n(1^-,2^-,3^+,\ldots,n^+) \\ &\quad= (-1)^{n+1}\sum_{\mathcal{P}(i_3,\ldots,i_n)}\frac{\langle 1\,2\rangle^6\,[1\,i_n]}{\langle 1\,i_n\rangle}\,\frac{[i_3\,i_4]}{\langle 2\,i_3\rangle\langle 2\,i_5\rangle\langle i_3\,i_4\rangle^2\langle i_4\,i_5\rangle}\left(\prod_{s=5}^{n-1}\frac{\langle 2|\,i_3+\cdots+i_{s-1}|i_s]}{\langle 2\,i_{s+1}\rangle\langle i_s\,i_{s+1}\rangle}\right). \end{aligned} \tag{2.15}$$

This is not quite the result (2.12). Note though that under exchange of $i_3$ and $i_4$, the product $\prod$ is invariant. Since we are summing over all permutations of the positive helicity lines $i_k$, we can include explicitly the $i_3 \leftrightarrow i_4$ permutation and divide by 2 to compensate for the overcounting. This allows us to rewrite (2.15) as

$$\begin{aligned} \frac{[i_3\,i_4]}{\langle 2\,i_3\rangle\langle 2\,i_5\rangle\langle i_3\,i_4\rangle^2\langle i_4\,i_5\rangle} &\to \frac{1}{2}\frac{[i_3\,i_4]}{\langle 2\,i_5\rangle\langle i_3\,i_4\rangle^2}\left(\frac{1}{\langle 2\,i_3\rangle\langle i_4\,i_5\rangle} - \frac{1}{\langle 2\,i_4\rangle\langle i_3\,i_5\rangle}\right) \\ &= \frac{1}{2}\frac{[i_3\,i_4]}{\langle 2\,i_3\rangle\langle 2\,i_4\rangle\langle i_3\,i_4\rangle\langle i_3\,i_5\rangle\langle i_4\,i_5\rangle}\,, \end{aligned}$$

by the Schouten identity. This gives (2.12) exactly.

## 3. BGK as (gauge theory)$^2$

The authors of [16] presented the BGK formula in a simpler form, which we write here as

$$M_n = -\langle a\,b\rangle^8 \sum_{\mathcal{P}(i_4,\ldots,i_n)} \frac{\prod_{s=4}^{n}\langle n|\,2+i_4+i_5+\cdots+i_{s-1}|\,i_s]}{\langle 1\,i_n\rangle\langle 1\,n\rangle^2\langle 2\,n\rangle^2\langle 1\,2\rangle\langle 2\,i_4\rangle\langle i_n\,n\rangle\big(\prod_{s=4}^{n-1}\langle i_s\,i_{s+1}\rangle\langle i_s\,n\rangle\big)}\,. \tag{3.1}$$

The external lines are $(1^+,2^+,\ldots,a^-,\ldots,b^-,\ldots,n^+)$ and the permutation sum $\mathcal{P}(i_4,\ldots,i_n)$ is over momentum labels $\{3,4,\ldots,n-1\}$.

The formula (1.6) is a simple rewriting of (3.1). First we relabel the external legs to the effect of interchanging $p_2$ and $p_n$. Then we select the negative helicity lines to be $a = 1$ and $b = 2$, and we introduce $i_3 = n$. Finally we rewrite the products in (3.1) to explicitly include the $A_n^2$ factor. The result is the formula (1.6). It is clear that by an initial relabeling of the external lines, the distinguished line $i_3$ could have been any one of the positive helicity lines.

The original BGK formula [17] can also be rewritten as a sum over gluon amplitudes squared, but we have chosen to work with (3.1) in order to display the form which most closely resembles our formula (1.4).

## 4. A modest non-MHV result

Loop amplitudes in gravity and supergravity require more than MHV tree amplitudes as input. For example the non-MHV amplitude[2] $M_6(1^-,2^-,3^-,4^+,5^+,6^+)$ was needed in the 3-loop calculation of [9]. Thus it would be of both practical and intrinsic interest to extend the treatment of recursion relations in section 2 to non-MHV amplitudes. Unfortunately the non-MHV sector is more complicated for both gluons and gravitons. Our results to date are limited to a new expression[3] for the anti-MHV amplitude $M_5(1^-,2^-,3^-,4^+,5^+)$ involving a sum over squares of gluon $A_5$'s. Of course, this amplitude is the complex conjugate of the MHV $M_5(1^+,2^+,3^+,4^-,5^-)$, and this fact provides a check which the formula obtained below satisfies. We present our formula with few details as an indication of the complications encountered in the non-MHV sector.

The relevant graviton recursion relation, obtained using a $[2,1\rangle$ shift, is

$$\begin{aligned} M_5(1^-,2^-,3^-,4^+,5^+) = &\left\{M_4(\hat{1}^-,3^-,P^+_{\hat{2}4},5^+)\,\frac{1}{s_{24}}\,M_3(-P^-_{\hat{2}4},\hat{2}^-,4^+) + (4\leftrightarrow 5)\right\} \\ &+M_4(\hat{1}^-,P^-_{\hat{2}3},4^+,5^+)\,\frac{1}{s_{23}}\,M_3(-P^+_{\hat{2}3},\hat{2}^-,3^-) \end{aligned} \tag{4.1}$$

Since the right side involves only 3- and 4-point functions we can insert the results (2.6) and (2.9), with conjugation and shifts as appropriate. The result is a sum of terms involving products $(A_4\,A_3)^2$ for various configurations of momenta. The strategy of section 2 suggests that we use gauge theory recursion relations to replace these products with $(A_5)^2$. However this is tricky because the recursion relation for one of the needed orderings of external gluons has two terms[4]

$$\begin{aligned} A_5(1^-,3^-,2^-,4^+,5^+) = &-A_4(\hat{1}^-,P^-_{\hat{2}3},4^+,5^+)\frac{1}{s_{23}}A_3(-P^+_{\hat{2}3},\hat{2}^-,3^-) \\ &+A_4(\hat{1}^-,3^-,P^+_{\hat{2}4},5^+)\frac{1}{s_{24}}A_3(-P^-_{\hat{2}4},\hat{2}^-,4^+)\,. \end{aligned} \tag{4.2}$$

Nevertheless we use (4.2) and the one-term recursion relations which hold for other order-

[2]Recursion relations were used in [8] to obtain a spinor helicity formula for this amplitude.
[3]A spinor helicity formula was given earlier in [19].
[4]The minus sign is required because of anti-cyclic ordering in the first term.



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ings to derive the following representation:

$$M_5(1^-,2^-,3^-,4^+,5^+) = \Big\{ s_{24}\, s_{\hat{1}5}^{z=z_{24}} \Big[A_5(1^-,2^-,3^-,4^+,5^+) + A_5(1^-,3^-,2^-,4^+,5^+)\Big]^2$$
$$+ s_{24}\, s_{35}\, A_5(3^-,1^-,2^-,4^+,5^+)^2$$
$$+ s_{23}\, s_{\hat{1}5}^{z=z_{23}}\, A_5(1^-,2^-,3^-,4^+,5^+)^2 \Big\} + (4 \leftrightarrow 5)\,, \tag{4.3}$$

which essentially does express the graviton $\overline{\text{MHV}}$ amplitude in terms of squares of $\overline{\text{MHV}}$ gluon amplitudes. Readers with good eyesight will notice that the invariant $s_{\hat{1}5}$ contains a shift to be evaluated at the appropriate poles,

$$P_{\hat{2}4}^2 = 0 \qquad \rightarrow \qquad s_{\hat{1}5}^{z=z_{24}} = \frac{\langle 35\rangle[15][34]}{[14]}\,, \tag{4.4}$$

$$P_{\hat{2}3}^2 = 0 \qquad \rightarrow \qquad s_{\hat{1}5}^{z=z_{23}} = -\frac{\langle 45\rangle[15][34]}{[13]}\,. \tag{4.5}$$

These results are used in the first and third line of (4.3), respectively.

## 5. Discussion

The formulas (1.4) and (1.6) express graviton MHV amplitudes $M_n$ as sums of gluon MHV amplitudes $A_n$ squared. This is a first step towards obtaining general-$n$ KLT-like relations from field theory. We have proven our formula (1.4) by induction using recursion relations. The fact that the BGK formula can be written in a very similar way (1.6) should facilitate an analytic proof of the BGK formula.

It was noted in [16] that under a $(-,-)$-shift the BGK formula (3.1) behaves as $z^{-2}$ for large $z$. Our rewriting (1.6) of (3.1) clearly exhibits this property too, and it also makes it manifest that, for this type of shift, the large $z$-behavior of $M_n$ is identical to that of $A_n^2$. On the other hand, our formula (1.4) has naively a leading $z^{-1}$ fall-off. We have checked numerically up to $n = 11$ that this leading term vanishes. This is an indication of the redundancy of the $(n-2)$ extra permutations in (1.4) compared with (1.6).

In the proof of (1.4), we first used the gravity recursion relations to express $M_n$ in terms of $M_3$ and $M_{n-1}$ and then the inductive assumption to get from $M_{n-1}$ to (sum of) $A_{n-1}^2$. A very useful step was then to use that the gauge theory recursion relations only contained one term, so that one could replace $A_3^2\, A_{n-1}^2$ by $s^2\, A_n^2$. It is not clear that one can generalize this step to non-MHV, since as we illustrated in section 3, the gauge recursion relations will contain several terms. Beyond $n = 5$ the $(-,-)$ shift does not seem to make the step $A_k^2 A_{n-k+2}^2 \rightarrow s^2\, A_n^2$ possible.

Tree amplitudes play an important role in loop calculations, and our work is a step towards deriving useful relations of the form $M_n = \sum A_n^2$ from field theory.

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